Logic, Reasoning, and Persuasion, Week 8-2 Handout

1 | Connectives

If A and B are statements, we can make more complicated statements out of them. There are a few standard ways to do this:

1. AND, written & or $\wedge$.
   (a) Given statements A and B, you can write $A\&B$ for "A and B."
   (b) $A\&B$ is a true statement if *both* A and B are true statements. It is false otherwise.
2. OR, written $\vee$.
   (a) Given statements A and B, you can write $A \vee B$ for "Either A or B (or both)."
   (b) $A \vee B$ is a true statement if *either* of A or B are true statements, or both are true. It is false otherwise.
3. NOT, written $\neg$ or $\sim$.
   (a) Given a statement A, you can write $\neg A$ for "it is not the case that A."
   (b) $\neg A$ is true if A is false, and is false if A is true.

We can write down these conditions in tables, called "truth tables," for reference.

A	$\neg A$
T	F
F	T

A	B	$A\&B$
T	T	T
T	F	F
F	T	F
F	F	F

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Here's how to use the truth tables: if we know A is true and B is false and want to know whether $A \vee B$ is true, we go to the truth table for $A \vee B$, and find the row where A is true and B is false:

	A	B	$A \vee B$
	T	T	T
$\rightarrow$	T	F	T
	F	T	T
	F	F	F

In this row $A \vee B$ is true. So if A is true and B is false, then $A \vee B$ is true.

Exercise 1: Let A = Ananya is working and B = Bernard is working. Suppose that Ananya is working, but Bernard is not. For each of the following, is it true or false?

1. A
2. B
3. $A \vee B$
4. $A\&B$
5. $\neg B$
6. $\neg A$

2 | Probabilities in Truth Tables

If you have just one statement A, and you're asking whether it's true or false, there are two possibilities: Either A is true or A is false (so $\neg A$ is true), and exactly one of them is true (since A and $\neg A$ can't both be true. If we have some level of confidence that A is true and some level of confidence that $\neg A$ is true, we can write this in a modified truth table. If there is a 0.4 probability that A is true, and 0.6 that A is false, then:

A		$Prob$
T	$Pr(A)$	0.4
F	$Pr(\neg A)$	0.6

The numbers in the second column will always add up to one. Why is this? Intuitively, it's because we are certain that either A is true or A is false, so by the second axiom, $Pr(\text{either } A \text{ is true or } A \text{ is false}) = 1$.

When we have two statements A and B, and we're asking whether they are true or false, there are four possibilities:

1. Both A and B are true
2. A is true, but B is not
3. B is true, but A is not
4. Neither A nor B are true

These four possibilities correspond to the four rows of the truth tables for A and B. So for example if we think all four options are equally likely, then:

A	B		$Prob$
T	T	$Pr(A\&B)$	0.25
T	F	$Pr(A\&\neg B)$	0.25
F	T	$Pr(\neg A\&B)$	0.25
F	F	$Pr(\neg A\&\neg B)$	0.25

Notice again that the numbers in the column add up to 1. Again, this is intuitively because exactly one of these four possibilities is true, so we are certain that *one* of them will be true.

3 | Calculating Probabilities with Truth Tables

Suppose we have two propositions, A and B, and the following probabilities for the possibilities:

A	B		$Prob$
T	T	$Pr(A\&B)$	0.2
T	F	$Pr(A\&\neg B)$	0.3
F	T	$Pr(\neg A\&B)$	0.4
F	F	$Pr(\neg A\&\neg B)$	0.1

To get the probabilities of these four possibilities, we can just read off that single row. But suppose we want to know the probability that $A \vee B$ is true? That is not one of the rows. Here's what we can do. We'll bring in the truth-table for $A \vee B$:

A	B		$A \vee B$	$Prob$
T	T	$Pr(A\&B)$	**T**	0.2
T	F	$Pr(A\&\neg B)$	**T**	0.3
F	T	$Pr(\neg A\&B)$	**T**	0.4
F	F	$Pr(\neg A\&\neg B)$	F	0.1

Then we'll add up the numbers in all the rows where $A \vee B$ has a T (bolded): 0.2 + 0.3 + 0.4 = 0.9. So $Pr(A \vee B) = 0.9$.

Exercise:
1. What is $Pr(\neg A)$ and $Pr(\neg B)$?
2. What is $Pr(\neg(A \vee B))$?

4 | Calculating Conditional Probabilities

What is the probability $Pr(A \mid B)$? There is no statement corresponding to $A \mid B$. Instead, we need a rule to calculate conditional probabilities:

Calculating Conditional Probability

Suppose you have the probability truth table for A and B and you want to know $Pr(B \mid A)$.
1. First, **suppose** that A is true: cross out the rows in which A is false, so only the ones in which A are *true* Add all the remaining rows: this is $Pr(A)$.
2. Second, add all the remaining rows where B is true. This is $Pr(B\&A)$ (since in these rows, both A and B are true).
3. Finally, divide the number from part (2) by the number in part (1) to calculate $Pr(B \mid A)$:

$$Pr(B \mid A) = \frac{Pr(B\&A)}{Pr(A)}.$$

Example:

Step 1: $Pr(A)$

A	B	$Prob$
$\rightarrow$ **T**	T	0.2
$\rightarrow$ **T**	F	0.3
$\cancel{F}$	$\cancel{T}$	~~0.4~~
$\cancel{F}$	$\cancel{F}$	~~0.1~~

$Pr(A) = 0.2 + 0.3 = 0.5$

Step 2: $Pr(A\&B)$

A	B	$Prob$
$\rightarrow$ **T**	**T**	0.2
T	F	0.3
$\cancel{F}$	$\cancel{T}$	~~0.4~~
$\cancel{F}$	$\cancel{F}$	~~0.1~~

$Pr(A\&B) = 0.2$

Step 3: $Pr(B \mid A)$

$$Pr(B \mid A) = \frac{Pr(B\&A)}{Pr(A)}$$

$$Pr(B \mid A) = \frac{0.2}{0.5} = 0.4.$$

Exercise: for the same table from the example, find $Pr(A \mid B)$.

For the probability truth table below, find $Pr(A \mid B)$ and $Pr(B \mid A)$.

A	B		$Prob$
T	T	$Pr(A\&B)$	0.5
T	F	$Pr(A\&\neg B)$	0.3
F	T	$Pr(\neg A\&B)$	0.1
F	F	$Pr(\neg A\&\neg B)$	0.1

Suppose we know the following about Ananya and Bernard:

1. There is a 10% chance that both will be in the office.
2. There is a 30% chance that only Ananya is in the office, and a 30% chance that only Bernard is in the office.
3. There is a 30% chance that neither of them are in the office.

Calculate $Pr(\text{Bob is in} \mid \text{Ananya is in})$ and $Pr(\text{Ananya is in} \mid \text{Bob is in})$.

5 | Back to the Axioms

Intuitively, when there are possibilities that never happen at the same time, it makes sense to add up the probabilities to figure out the probability that *one* of them happens or the other one happens. This can be formalized in the final axiom of probability:

Probability Axiom 3: Additivity

PA3 If A and B are never true at the same time, then $Pr(A \vee B) = Pr(A) + Pr(B)$.

With this last axiom in hand, let's review the other axioms. I'll admit I simplified things a bit last time: time for the full truth.

The Three Probability Axioms

PA1 **Nonnegativity**: For any statement A, $Pr(A) \geq 0$. *Intuitively: your confidence in any possibility should never be negative.*

PA2 **Certainty**: For any statement A, if you're certain that A is true, then $Pr(A) = 1$ (and if you're certain that A is false, then $Pr(A) = 0$).

PA3 **Additivity** If A and B are never true at the same time, then $Pr(A \vee B) = Pr(A) + Pr(B)$.