Logic, Reasoning, and Persuasion, Week 8-2 Handout

1 | COVID-19 Tests

Suppose you take a COVID-19 test with the following accuracy:

1. On people who **do** have COVID-19, it gives a *positive* result 80% of the time, and gives a negative result the other 20% of the time.
2. On people who **don't** have COVID-19, it gives a *negative* result 80% of the time, and gives a positive result the other 20% of the time.

Here a *positive* result is a result that says you have COVID-19, and a *negative* result is a result that says you do not have COVID-19.

> **Exercise 1:** You take the test, and it comes up positive.
> 1. How confident should you be that you have COVID-19?
> 2. Now suppose that 5% of your community has COVID-19, and you don't have any particular information about yourself that makes you think you're more or less likely than the average person to have it.
> 3. Now suppose that 50% of your community has COVID-19, and you still don't have particular information about yourself.

2 | Probability

I throw a six sided fair die, and I don't show it to you. How likely is it that it landed on 6?

Almost all the time, there a bunch of things about the world that we're *unsure* about. Sometimes, we can be more or less confident about these things. In these cases, we can say we have estimates of the **probability** that these things are true. For instance, the weather forecast can be 67% confident it will rain. You can be 1/2 confident that a fair coin will land hands, or 25% confident that the bus will come on time.

> **Writing Probabilities**
>
> Let H be some statement (H for "Hypothesis"). Let's use $P(H)$ to mean "the probability that H is true."

These probabilities have numbers: 67%, 1/2, 25%. How do we determine these numbers? Well, to make the math easier, we'll use two rules:

> **Probability Axioms 1 and 2**
>
> PA1 If you're certain, or 100% sure, that H is *true*, then $P(H) = 1$.
>
> PA2 If you're certain, or 100% sure, that H is *false*, then $P(H) = 0$.[a]
>
> *a.* Notice you could also say then that you are 0% sure that H is true. But this is a bit awkward.

Exercise 2:
1. What is $P(\text{we are at Rutgers right now})$?
2. What is $P(\text{the stock market crashes next month})$?
3. What is $P(\text{There is nobody in this room})$?

2.1 | *Counting*

When something has a bunch of equally likely outcomes, like the throw of a fair die, we can say the probability of some outcomes $H1$, $H2$, is just the number of outcomes divided by the *total* number of possible outcomes. For instance, the probability that a fair die lands on 4 or 6 is just 2 (2 possibilities) divided by 6 (the total possible outcomes): $2/6$.

Counting Equally Likely Outcomes

If there are bunch of equally likely outcomes and you want to know how likely some of them are, then you can calculate like this:

$$P(\text{an outcome you are counting happens}) = \frac{\#\text{outcomes you are counting}}{\#\text{all outcomes}}.$$

Exercise 3: I throw a six sided fair die, and I don't show it to you.
1. What is $P(\text{the die landed on 6})$?
2. What is $P(\text{the die landed even})$?
3. What is $P(\text{the die landed even and landed on 5})$?

3 | Conditional Probability

Suppose I told you the die landed on an even number (2,4,6). Then what would be the probability that it landed on 6?
When we think about probabilities, we can think about the probability that certain statements are true *supposing* other statements are true. This is called *conditional probability* and it's a basic element of the theory of inductive reasoning.

Conditional Probability

Let E be another statement (E for Evidence). Let $P(H \mid E)$ mean the probability that H is true *supposing* that E is true.

Exercise 4
1. I throw a six sided fair die, and I don't show it to you.
   (a) What is $P(\text{the die landed on 6} \mid \text{the die landed even})$?
   (b) What is $P(\text{the die landed even} \mid \text{the die landed even})$?
   (c) What is $P(\text{the die landed odd} \mid \text{the die landed even})$?
   (d) What is $P(\text{the die landed even} \mid \text{the die landed prime})$?

4 | Back to COVID-19 Tests

Remember the setup for COVID-19 tests. We have a test with the following accuracy:

1. On people who **do** have COVID-19, it gives a *positive* result 80% of the time, and gives a negative result the other 20% of the time.
2. On people who **don't** have COVID-19, it gives a *negative* result 80% of the time, and gives a positive result the other 20% of the time.

Exercise 5: Suppose that 5% of the community you're in has COVID-19, and you don't have special reason to think you're more or less likely than the average person to be sick.

1. What is $P(\text{you test positive} \mid \text{you are sick})$?
2. What is $P(\text{you test positive} \mid \text{you are not sick})$?
3. Before you take any test, what is $P(\text{you are sick})$?
4. What is $P(\text{you test positive})$?

To solve the last part of Exercise 5, let's go back to numbers. If you know nothing in particular about yourself, then one way to get $P(\text{you test positive})$ is to count the number of people who test positive in your community:

$$P(\text{you test positive}) = \frac{\#\text{people who test positive}}{\#\text{people}}.$$

We'll suppose there are 100 people in the community. (This number won't end up mattering, as we'll see next time.)

The denominator (bottom of the fraction) is easy: $\#\text{people} = 100$. The numerator (top of the fraction) is less obvious. Here are things we know:

- The number of people (100).
- The number of sick people (5).
- The number of healthy people (95).
- Proportion of sick people who test positive (0.8, or 80%).
- Proportion of healthy people who test positive (0.2, or 20%).

We *don't* know the number of people who test positive. However, we can calculate it:

$$\#\text{people who test positive} = \#\text{healthy people who test positive} + \#\text{sick people who test positive}$$

And then we calculate how many healthy and sick people test positive:

$$\begin{aligned}\#\text{healthy people who test positive} &= \#\text{healthy people } \times \text{proportion of healthy people who test positive}\\ &= 95 \times 0.2 = 18.\\ \#\text{sick people who test positive} &= \#\text{sick people } \times \text{proportion of sick people who test positive}\\ &= 5 \times 0.8 = 4.\end{aligned}$$

So #people who test positive $= 18\ \text{healthy} + 4\ \text{sick} = 22$.
Thus $P(\text{you test positive}) = \frac{22}{100} = 0.22$.

Exercise 6: What is $P(\text{you test positive})$ if not 5%, but 50% of the community has COVID-19? To figure this out,

1. Calculate the number of healthy people and the number of sick people.
2. Then use (1) to calculate the number of healthy who test positive and the number of sick people who test positive.
3. Then use (2) to calculate the number of people who test positive.
4. Then use (3) to calculate $P(\text{you test positive})$.

5 | So, Are You Sick?

Our main goal was to figure out the probability that you are sick, supposing that (or given that) you test positive. Now we have the means to do that.

Exercise 7: What is $P(\text{you are sick} \mid \text{you test positive})$ if 5% of the community has COVID-19?

When you didn't know anything about yourself to make you think you were more or less likely than the average person to be sick, you could just calculate the probability that you were sick by finding the proportion of sick people in the community:

$$P(\text{you are sick}) = \frac{\#\text{sick people}}{\#\text{people}}.$$

But if you test positive, you know that you are one of the people in your community who tested positive. So you can calculate the probability that you were sick by finding the proportion of sick people among those people who tested positive:

$$P(\text{you are sick} \mid \text{you tested positive}) = \frac{\#\text{sick people who tested positive}}{\#\text{people who tested positive}} = \frac{4}{22} \approx 0.18.$$

Exercise 8: What is $P(\text{you are sick} \mid \text{you test positive})$ if 50% of the community has COVID-19?

6 | Updating Your Opinions on Evidence

Inductive Reasoning is about *updating your opinions on evidence*:

$$\text{have opinion } P_{old}(H) \longrightarrow \text{get evidence } E \longrightarrow \text{update to opinion } P_{new}(H)$$

For the COVID-19 test case, the update goes like this:

$$P_{old}(\text{sick}) = 0.05 \quad \longrightarrow \quad E = \text{tested positive} \quad \longrightarrow \quad P_{new}(H) \approx 0.18.$$

The rest of the course will be about just how to go from P_{old} to P_{new} when you learn new evidence.