Logic, Reasoning, and Persuasion, Week 10-2 Handout

1 | Understanding Total Probability

Last time we learned the Law of Total Probability:

The Law of Total Probability

For any evidence statement E and possible hypotheses $H_1, H_2, \ldots, H_n$, where exactly one hypothesis is true, the probability of E is given by:

$$\begin{aligned}\Pr(E) = \quad & \Pr(E \mid H_1)\Pr(H_1) \\ & + \Pr(E \mid H_2)\Pr(H_2) \\ & \ldots \\ & + \Pr(E \mid H_n)\Pr(H_n).\end{aligned}$$

where $\Pr(E \mid H)\Pr(H)$ is shorthand for multiplication: $\Pr(E \mid H) \times \Pr(H)$.

Example: Ariel, Branden, and Carissa came over yesterday to hang out, and one of them left their water bottle. You don't know whose it is, and you're trying to see whose it's most likely to be. Your hypotheses are as follows:

H_1 = It was Ariel, H_2 = It was Branden, H_3 = It was Carissa.

Before examining the bottle, you think Ariel is twice as likely as Branden or Carissa to have left the bottle (she's always thirsty). So

$$\Pr(H_1) = 1/2, \quad \Pr(H_2) = 1/4, \quad \Pr(H_3) = 1/4.$$

You examine the bottle and find a sticker for the debate club:

E = You found a sticker for the debate club.

Ariel has never been part of the debate club, so you think it's very unlikely she would put a sticker on her bottle. Branden and Carissa are both in the debate club, but Carissa is much more into it. So you think that Carissa is much more likely than Branden to put a sticker on her bottle. You estimate:

1. $\Pr(E \mid H_1) = 1/10$ *(There's a 10% chance Ariel would put a sticker on her bottle).*
2. $\Pr(E \mid H_2) = 1/4$ *(There's a 25% chance Branden would put a sticker on his bottle).*
3. $\Pr(E \mid H_3) = 3/4$ *(There's a 75% chance Carissa would put a sticker on her bottle).*

By Bayes rule, you know you can figure out how likely it is to be each person's bottle:

$$\Pr(H_i \mid E) = \Pr(H_i)\frac{\Pr(E \mid H_i)}{\Pr(E)}.$$

The problem is that you don't know $\Pr(E)$: you don't know how probable it was that you would find a sticker on the bottle in the first place.

Total probability helps you figure this out. You reason: "there are exactly three ways for me to have found the sticker":

1. I found the sticker and it was Ariel's bottle: $E\&H_1$.
2. I found the sticker and it was Branden's bottle: $E\&H_2$.
3. I found the sticker and it was Carissa's bottle: $E\&H_2$.

Since these never happen at the same time, the third axiom of probability tells us that we can get the probability of E by adding these probabilities together:

$$\Pr(E) = \Pr(E\&H_1) + \Pr(E\&H_2) + \Pr(E\&H_3).$$

Then the ratio formula tells us that the probability that you find the sticker *and* that the bottle is Ariel's is just the probability that you find the sticker *given that* the bottle is Ariel's, times the probability that the bottle is Ariel's, and likewise for Branden and Carissa. Substituting these in:

$$\Pr(E) = \Pr(E \mid H_1)\Pr(H_1) + \Pr(E \mid H_2)\Pr(H_2) + \Pr(E \mid H_3)\Pr(H_3).$$

Since you know all these numbers, you can calculate $\Pr(E)$. And this calculation reveals that the probability that you get the evidence (find the sticker) is just the probability of each hypothesis H_1, H_2, H_3, multiplied by the probability that you get the evidence *given that hypothesis*, and then added all together. In a slogan:

> **Total Probability Slogan**: The probability that I learn E is just the probability I learn E supposing that different hypotheses are true, multiplied by how likely those hypotheses are to be true.

With this in hand, we can do the Bayes' Theorem calculation without issue.

1.1 | Back to Bayes

With total probability, we can understand Bayes' Theorem in another way. To find the probability of any hypothesis, given the evidence you learn, you first figure out what your hypotheses are, how likely they are, and how likely you are to get the evidence given the hypotheses. Then you write the sum in both the numerator and the denominator of a fraction:

$$\frac{\Pr(E)}{\Pr(E)} = \frac{\Pr(E \mid H_1)\Pr(H_1) + \Pr(E \mid H_2)\Pr(H_2) + \Pr(E \mid H_3)\Pr(H_3)}{\Pr(E \mid H_1)\Pr(H_1) + \Pr(E \mid H_2)\Pr(H_2) + \Pr(E \mid H_3)\Pr(H_3)}.$$

Then Bayes' Theorem allows you to figure out the probability of any particular hypothesis, given the evidence, by keeping *only* that hypothesis in the numerator:

$$\Pr(H_1 \mid E) = \frac{\Pr(E \mid H_1)\Pr(H_1)}{\Pr(E \mid H_1)\Pr(H_1) + \Pr(E \mid H_2)\Pr(H_2) + \Pr(E \mid H_3)\Pr(H_3)}$$

$$\Pr(H_2 \mid E) = \frac{\Pr(E \mid H_2)\Pr(H_2)}{\Pr(E \mid H_1)\Pr(H_1) + \Pr(E \mid H_2)\Pr(H_2) + \Pr(E \mid H_3)\Pr(H_3)}$$

$$\Pr(H_3 \mid E) = \frac{\Pr(E \mid H_3)\Pr(H_3)}{\Pr(E \mid H_1)\Pr(H_1) + \Pr(E \mid H_2)\Pr(H_2) + \Pr(E \mid H_3)\Pr(H_3)}.$$

2 | Evidence For

Let's say that a statement E is *evidence* for a hypothesis H if the probability of H is higher when we suppose E is true than when we don't.

Definition 1. E is evidence **for** H if

$$\Pr(H \mid E) > \Pr(H).$$

E is evidence **against** H if

$$\Pr(H \mid E) < \Pr(H).$$

Example: Let H = "The die comes up 6" and let E = "The die comes up heads". Is E evidence for H?

It turns out that when it comes to E being evidence for H, you can switch E and H! The *Evidence-For* lemma says that E is evidence for H exactly when getting E is more probable when you suppose that H is true than if you don't.

Lemma 1 (Evidence-For)**.** *E is evidence for H if and only if*

$$\Pr(E \mid H) > \Pr(E),$$

and E is evidence against H if and only if

$$\Pr(E \mid H) < \Pr(E).$$

This switch is helpful, for the same reason that Bayes' Theorem was helpful in the first place: you will often know $\Pr(E \mid H)$ but not know $\Pr(H \mid E)$.

Exercise 1:

1. Is finding a sticker on the bottle evidence for H_1?
   How about H_2 and H_3?
2. After you find the sticker on the bottle, who should you think is most likely the owner of the bottle?

We learn two things about evidence from the sticker case:

1. **Multiple Hypotheses**: One piece of evidence E can be evidence for multiple different hypotheses at the same time.
2. **Inconclusiveness**: A piece of evidence E can evidence for a hypothesis H without making H the most likely hypothesis.

Exercise 2:

1. If $\Pr(E \mid H) = 0.5$ and $\Pr(E) = 0.4$, is E evidence for or against H?
2. If $\Pr(E \mid H) = 0.5$, $\Pr(H) = 0.8$, and $Pr(E) = 0.9$, is E evidence for or against H?
3. If $\Pr(E \mid H) = 0.5$, $\Pr(H \mid E) = 0.7$, and $Pr(H) = 0.6$, is E evidence for or against H?

Exercise 3: Marbles in a Bag

I have a box with an half black marbles and half white marbles. I take a marble out randomly and place it in an cloth bag.

Q1 What is the probability that the marble in the bag is white?

(Carroll 1958) I take a white marble out of the box, and place it in the bag. I shake the bag, and randomly take one of the two marbles out. I look at the color: it is white.

Q2 Should you *increase* or *decrease* your confidence that the remaining marble in the bag is white, or keep it the same?

Q3 What is the probability that the remaining marble in the bag is white?

To help with **Q2** and **Q3**, let's ask some further questions. Let

$$E = \text{the marble I randomly take out is white.}$$

Call the first marble I put in marble one, and the second marble (the one I know to be white) marble two. Now let

$$WW = \text{marble one is white and marble two is white}$$

$$BW = \text{marble one is black and marble two is white.}$$

Based on the setup of the case, we know that exactly one of WW and BW is true.

1. **Step One**: If I randomly take one of the two marbles out, and learn that E (that the marble I took out is white), is this evidence for or against WW?
   (a) You can use the definition, and figure out if $\Pr(WW \mid E) > \Pr(WW)$.
   (b) Or use the *Evidence-For* lemma and figure out if $\Pr(E \mid WW) > \Pr(E)$.

   Then if E is evidence for (/against) WW, you should increase (/decrease) your confidence that the remaining marble is white.
2. **Step Two**: Use Bayes' Theorem to calculate $\Pr(WW \mid E)$:

$$\begin{aligned}\Pr(WW \mid E) &= \Pr(WW)\frac{\Pr(E \mid WW)}{\Pr(E)} \\ &= \frac{\Pr(E \mid WW)\Pr(WW)}{\Pr(E \mid WW)\Pr(WW) + \Pr(E \mid BW)\Pr(BW)}.\end{aligned}$$

More Exercises (optional)

1. Prove the Evidence-For lemma from Bayes' Theorem and the definition of "Evidence For."
2. After taking out the marble (which turned out to be white) from the bag , I get another bag and put a white marble in along with two black marbles. Now, what gives you a better chance of drawing a white marble? (Also Carroll (1958)).
   (a) Randomly choose one of the bags, and then draw a marble from that bag.
   (b) Pour the marbles into the same bag, and draw a marble from that bag.

   What are the chances in each case?
3. Show that if E and H are the same statement, then $\Pr(H \mid E) = 1$.